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1987 J. Phys. A: Math. Gen. 20 L1069

(http://iopscience.iop.org/0305-4470/20/16/006)

J. Phys. A: Math. Gen. **20** (1987) L1069–L1070. Printed in the UK

LETTER TO THE EDITOR

The anisotropic Heisenberg spin chain and the derivative non-linear Schrödinger equation

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Received 15 July 1987

Abstract. It is shown that the equation of motion for the classical continuous Heisenberg spin chain with uniaxial anisotropy is obtained as a reduction of the two-component derivative non-linear Schrödinger equation.

The key step in the solution of the Korteweg–de Vries equation was the transformation from the modified Korteweg–de Vries equation to the Korteweg–de Vries equation discovered by Miura (1968) and now bearing his name.

Here we will be concerned with Miura transformations to the non-linear Schrödinger (NLS) equation. The first of these, from the isotropic Heisenberg spin chain to the NLS, was found by Hasimoto (1972). This was generalised to a transformation from the uniaxially anisotropic Heisenberg spin chain (AHSC) to the NLS by Quispel and Capel (1982). Another Miura transformation, from the two-component derivative non-linear Schrödinger equation (DNLS) to the two-component NLS was found by Wadati and Sogo (1983). (Alternatively one can say they showed the gauge equivalence of the two-component DNLS and the two-component NLS.)

In this letter we generalise Wadati and Sogo's Miura transformation slightly and show that under an appropriate reduction it goes over into the one found by Quispel and Capel. It turns out that this is equivalent to a proof of the fact that the two-component DNLS contains the AHSC as a special non-trivial reduction.

Let u and v satisfy the two-component DNLS (Clarkson and Cosgrove 1987, van der Linden *et al* 1986):

$$\mathrm{i}u_t = u_{xx} - \mathrm{i}bu^2v_x + \tfrac{1}{2}b^2u^3v^2 + abu^2v \tag{1a}$$

$$-\mathrm{i}v_t = v_{xx} + \mathrm{i}bv^2u_x + \tfrac{1}{2}b^2v^3u^2 + abv^2u \tag{1b}$$

where, in general, a and b are complex constants. Then $\tilde{u}$ and $\tilde{v}$, defined by

$$\tilde{u} := u \tag{2a}$$

$$\tilde{v} := -\mathrm{i}v_x + \tfrac{1}{2}buv^2 + av \tag{2b}$$

satisfy the two-component NLS

$$\begin{aligned}\mathrm{i}\tilde{u}_t &= \tilde{u}_{xx} + b\tilde{u}^2\tilde{v}\\ -\mathrm{i}\tilde{v}_t &= \tilde{v}_{xx} + b\tilde{u}\tilde{v}^2\end{aligned} \tag{3}$$

(notice that (2b) is a Riccati equation in v).

We will now consider reductions. A simple reduction that is obviously allowed is to take $v = u^*$ (and a and b real). (1a) and (2a) then give

$$\mathrm{i}\tilde{u}_t = \tilde{u}_{xx} - \mathrm{i}b\tilde{u}^2\tilde{u}_x^* + \tfrac{1}{2}b^2\tilde{u}^3\tilde{u}^{*2} + ab\tilde{u}^2\tilde{u}^* \tag{4}$$

i.e. the two-component NLS contains the DNLS as a reduction. A somewhat more complicated reduction obtains imposing $\tilde{v} = \tilde{u}^*$ (and b real). From (2) we then obtain

$$u^* = -\mathrm{i}v_x + \tfrac{1}{2}buv^2 + av. \tag{5}$$

The question to be answered is whether (5) is consistent with (1).

From (5) and its complex conjugate we obtain

$$u^* = \frac{-\mathrm{i}v_x + \frac{1}{2}\mathrm{i}bv^2v_x^* + \frac{1}{2}abv^2v^* + av}{1 - \frac{1}{4}b^2|v|^4}. \tag{6}$$

Inserting (5) and (6) in (1) we obtain the equations that u and v satisfy:

$$\mathrm{i}u_t = u_{xx} + bu^2u^* \tag{7a}$$

$$\begin{aligned}-\mathrm{i}v_t = {}& \partial_x\left(\frac{\partial_x v - \frac{1}{2}bv^2\partial_x v^*}{1 - \frac{1}{4}b^2|v|^4}\right) - \tfrac{1}{2}bv^2\partial_x\left(\frac{\partial_x v^* - \frac{1}{2}bv^{*2}\partial_x v}{1 - \frac{1}{4}b^2|v|^4}\right)\\ & + \frac{bv}{1 - \frac{1}{4}b^2|v|^4}(b\partial_x v - \tfrac{1}{2}b^2v^2\partial_x v^*)(b\partial_x v^* - \tfrac{1}{2}b^2v^{*2}\partial_x v)\\ & + \frac{\mathrm{i}b(a + a^*)v^2\partial_x v^*}{(1 - \frac{1}{2}b|v|^2)^2} + \frac{\frac{1}{2}b^2(a + a^*)v^3v^{*2}}{(1 - \frac{1}{4}b^2|v|^4)^2} - \frac{baa^*v^2v^*}{(1 + \frac{1}{2}b|v|^2)^2}.\end{aligned} \tag{7b}$$

The only thing we still have to prove is that (7) is consistent with (6). Fortunately this was shown in Quispel and Capel (1983), where it was also shown that (7b) is equivalent to the AHSC:

$$\boldsymbol{S}_t = \boldsymbol{S} \times \boldsymbol{S}_{xx} + \boldsymbol{S} \times A S_z \boldsymbol{e}_z \qquad \boldsymbol{S} \cdot \boldsymbol{S} = 1. \tag{8}$$

We conclude that the two-component DNLS contains the NLS and the AHSC as reductions.

References

Clarkson P A and Cosgrove C M 1987 *J. Phys. A: Math. Gen.* **20** 2003-24

Hasimoto H 1972 *J. Fluid. Mech.* **51** 477-85

Miura R M 1968 *J. Math. Phys.* **9** 1202-4

Quispel G R W and Capel H W 1982 *Phys. Lett.* **88A** 371-4

—— 1983 *Physica* **117A** 76-102

van der Linden J, Nijhoff F W, Capel H W and Quispel G R W 1986 *Physica* **137A** 44-80

Wadati M and Sogo 1983 *J. Phys. Soc. Japan* **52** 394-8